

ON CONFORMAL BIHARMONIC IMMERSIONS

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ABSTRACT

This paper studies conformal biharmonic immersions. We first study the transformations of Jacobi operator and the bitension field under conformal change of metrics. We then obtain an invariant equation for a conformal biharmonic immersion of a surface into Euclidean 3-space. As applications, we construct a 2-parameter family of non-minimal conformal biharmonic immersions of cylinder into \mathbb{R}^3 and some examples of conformal biharmonic immersions of 4-dimensional Euclidean space into sphere and hyperbolic space thus provide many simple examples of proper biharmonic maps with rich geometric meanings. These suggest that there are abundant proper biharmonic maps in the family of conformal immersions. We also explore the relationship between biharmonicity and holomorphicity of conformal immersions of surfaces.

1. INTRODUCTION

This paper works on the smooth objects, so we assume that manifolds, maps, vector fields, etc, are smooth unless it is stated otherwise.

A biharmonic map is a map $\varphi : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds that is a critical point of the bienergy functional

$$E^2(\varphi, \Omega) = \frac{1}{2} \int_{\Omega} |\tau(\varphi)|^2 dx$$

for every compact subset Ω of M , where $\tau(\varphi) = \text{Trace}_g \nabla d\varphi$ is the tension field of φ . The Euler-Lagrange equation of this functional gives the biharmonic map equation ([10])

$$(1) \quad \tau^2(\varphi) := \text{Trace}_g(\nabla^\varphi \nabla^\varphi - \nabla_{\nabla_M}^\varphi) \tau(\varphi) - \text{Trace}_g R^N(d\varphi, \tau(\varphi)) d\varphi = 0,$$

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which states the fact that the map φ is biharmonic if and only if its bitension field $\tau^2(\varphi)$ vanishes identically. In the above equation we have used R^N to denote the curvature operator of (N, h) defined by

$$R^N(X, Y)Z = [\nabla_X^N, \nabla_Y^N]Z - \nabla_{[X, Y]}^N Z.$$

Harmonic maps are clearly biharmonic, so it is more interesting to study *proper* (meaning non-harmonic) biharmonic maps as far as one seeks to pursue a new study. However, apart from the maps between Euclidean spaces defined by polynomials of degree less than four (a class of maps that seems so wild to exhibit any characteristic property) not many examples of proper biharmonic maps between Riemannian manifolds have been found (see, e.g., [15], [14], [17], and the bibliography of biharmonic maps [12]). So, currently, one priority and a practical thing to do seems to be finding more examples of proper biharmonic maps between certain model spaces or studying biharmonic maps under some geometric constraints. For example, one can study biharmonic isometric immersions which lead to the concept of biharmonic submanifolds (see e.g., [11], [6], [7], [4], [5], [15] and [3]); one can also study, as in [1], [2], [14], horizontally weakly conformal biharmonic maps which generalize both the notion of harmonic morphisms (maps that are both horizontally weakly conformal and harmonic) and that of biharmonic morphisms (maps that are horizontally weakly conformal biharmonic with other constraints, see [16], [13], [17], and [14] for details).

The interesting link between harmonicity and conformality has a long history. It was known to Weierstrass that a conformal immersion $\varphi : M^2 \longrightarrow \mathbb{R}^3$ is harmonic if and only if $\varphi(M)$ is a minimal submanifold of \mathbb{R}^3 . It is also well known that **conformal harmonic immersions** of surfaces are precisely **conformal minimal immersions** of surfaces of which there has been a rich theory exhibiting a beautiful interplay among geometry, topology, and real and complex analysis. So it would be interesting to know if we can generalize (or use the tools of) the theory on conformal minimal immersions to conformal biharmonic immersions. On the other hand, Jiang and Chen-Ishikawa independently proved that an isometric immersion $\varphi : M^2 \longrightarrow \mathbb{R}^3$ is biharmonic if and only if φ is harmonic. It would also be interesting to know whether this result can be generalized to the case of conformal biharmonic immersions. Motivated by these, we study conformal biharmonic immersions in this paper. First, we study the transformations of Jacobi operator and bitension field under conformal change of metrics. We then obtain an invariant equation for a conformal biharmonic immersion of a surface into Euclidean 3-space, and using this, we construct a 2-parameter family of non-minimal conformal biharmonic immersions of cylinder into \mathbb{R}^3 and some examples

of conformal immersions of 4-dimensional Euclidean space into sphere and hyperbolic space, thus provide many simple examples of proper biharmonic maps with rich geometric meanings. We also explore Weierstrass type representations for conformal biharmonic immersions.

2. JACOBI OPERATORS AND THE BITENSION FIELDS UNDER CONFORMAL CHANGE OF METRICS

For a map $\varphi : (M^m, g) \longrightarrow (N^n, h)$, the Jacobi operator is defined as

$$(2) \quad J_g^\varphi(X) = -\{\text{Trace}_g(\nabla^\varphi \nabla^\varphi - \nabla_{\nabla_M}^\varphi)X - \text{Trace}_g R^N(d\varphi, X)d\varphi\}$$

for any vector field X along the map φ . Thus, by (1) and (2), the relationship between the Jacobi operator and the bitension field of φ is explained by $J_g^\varphi(\tau(\varphi)) = -\tau^2(\varphi)$.

Theorem 1. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a map. Then, under the conformal change of metrics $\bar{g} = F^{-2}g$, we have*

(I) *the transformation of the Jacobi operators J_g^φ and $J_{\bar{g}}^\varphi$ of φ is given by*

$$(3) \quad J_{\bar{g}}^\varphi(X) = F^2 J_g^\varphi(X) + F^2(m-2)\nabla_{\text{grad ln } F}^\varphi X,$$

and

(II) *the transformation of the bitension fields $\tau^2(\varphi, g)$ and $\tau^2(\varphi, \bar{g})$ of φ is given by*

$$(4) \quad \begin{aligned} \tau^2(\varphi, \bar{g}) = & F^4\{\tau^2(\varphi, g) + (m-2)J_g^\varphi(d\varphi(\text{grad ln } F)) \\ & + 2(\Delta \ln F - (m-4)|\text{grad ln } F|^2)\tau(\varphi, g) - (m-6)\nabla_{\text{grad ln } F}^\varphi \tau(\varphi, g) \\ & - 2(m-2)(\Delta \ln F - (m-4)|\text{grad ln } F|^2)d\varphi(\text{grad ln } F) \\ & + (m-2)(m-6)\nabla_{\text{grad ln } F}^\varphi d\varphi(\text{grad ln } F)\}, \end{aligned}$$

where grad and Δ denote the gradient and the Laplacian taken with respect to the metric g .

Proof. Choose a local orthonormal frames $\{e_i\}$ with respect to g on M , then $\{\bar{e}_i = F e_i\}$ is a local orthonormal frames with respect to \bar{g} .

A direct computation gives the transformation of the tension fields under the conformal change of a metric as

$$\tau(\varphi, \bar{g}) = F^2\{\tau(\varphi, g) - (m-2)d\varphi(\text{grad ln } F)\}.$$

Also, a straightforward computation (see, e.g., [1]) yields

$$(5) \quad \begin{aligned} & \text{Trace}_{\bar{g}}(\nabla^\varphi \nabla^\varphi - \nabla_{\bar{\nabla}_M}^\varphi)X \\ &= F^2 \{ \text{Trace}_g(\nabla^\varphi \nabla^\varphi - \nabla_{\nabla_M}^\varphi)X - (m-2) \nabla_{\text{grad ln } F}^\varphi X \}. \end{aligned}$$

On the other hand,

$$(6) \quad \begin{aligned} \text{Trace}_{\bar{g}} R^N(\text{d}\varphi, X) \text{d}\varphi &= \sum_{i=1}^m R^N(\text{d}\varphi(Fe_i), X) \text{d}\varphi(Fe_i) \\ &= F^2 \text{Trace}_g R^N(\text{d}\varphi, X) \text{d}\varphi. \end{aligned}$$

Using Equations (5) and (6) we have

$$\begin{aligned} -J_g^\varphi(X) &= \text{Trace}_{\bar{g}}(\nabla^\varphi \nabla^\varphi - \nabla_{\bar{\nabla}_M}^\varphi)X - \text{Trace}_{\bar{g}} R^N(\text{d}\varphi, X) \text{d}\varphi \\ &= F^2 \{ \text{Trace}_g(\nabla^\varphi \nabla^\varphi - \nabla_{\nabla_M}^\varphi)X - \text{Trace}_g R^N(\text{d}\varphi, X) \text{d}\varphi \} \\ &\quad - (m-2) F^2 \nabla_{\text{grad ln } F}^\varphi X \\ &= -F^2 J_g^\varphi(X) - (m-2) F^2 \nabla_{\text{grad ln } F}^\varphi X, \end{aligned}$$

from which we obtain part (I) of the theorem.

To prove the second part of the Theorem, we compute

$$\begin{aligned} J^\varphi(fX) &= -\{ \text{Trace}_g(\nabla^\varphi \nabla^\varphi - \nabla_{\nabla_M}^\varphi)(fX) - \text{Trace}_g R^N(\text{d}\varphi, fX) \text{d}\varphi \} \\ (7) \quad &= fJ^\varphi(X) - (\Delta f)X - 2\nabla_{\text{grad } f}^\varphi X, \\ (8) \quad &\Delta F^2 = 2F^2 \Delta \ln F + 4F^2 |\text{grad ln } F|^2, \\ (9) \quad &\nabla_{\text{grad } F^2}^\varphi \text{d}\varphi(\text{grad ln } F) = 2F^2 \nabla_{\text{grad ln } F}^\varphi \text{d}\varphi(\text{grad ln } F), \\ (10) \quad &\nabla_{\text{grad ln } F}^\varphi (F^2 \text{d}\varphi(\text{grad ln } F)) = 2F^2 |\text{grad ln } F|^2 \text{d}\varphi(\text{grad ln } F) \\ &\quad + F^2 \nabla_{\text{grad ln } F}^\varphi \text{d}\varphi(\text{grad ln } F), \\ (11) \quad &\nabla_{\text{grad } F^2}^\varphi \tau(\varphi, g) = 2F^2 \nabla_{\text{grad ln } F}^\varphi \tau(\varphi, g), \text{ and} \\ (12) \quad &\nabla_{\text{grad ln } F}^\varphi (F^2 \tau(\varphi, g)) = 2F^2 |\text{grad ln } F|^2 \tau(\varphi, g) + F^2 \nabla_{\text{grad ln } F}^\varphi \tau(\varphi, g). \end{aligned}$$

Substituting $X = \tau(\varphi, \bar{g}) = F^2\{\tau(\varphi, g) - (m-2)d\varphi(\text{grad } \ln F)\}$ into (3) and using Equations (7)–(12) we have

$$\begin{aligned}
& \tau^2(\varphi, \bar{g}) = -J_g^\varphi(\tau(\varphi, \bar{g})) \\
& = -F^2\{J_g^\varphi(F^2\tau(\varphi, g) - (m-2)F^2d\varphi(\text{grad } \ln F)) \\
& \quad + (m-2)\nabla_{\text{grad } \ln F}^\varphi(F^2\tau(\varphi, g) - (m-2)F^2d\varphi(\text{grad } \ln F))\} \\
& = -F^2\{F^2J_g^\varphi(\tau(\varphi, g)) - (\Delta F^2)\tau(\varphi, g) - 2\nabla_{\text{grad } F^2}^\varphi\tau(\varphi, g) \\
& \quad - (m-2)F^2J_g^\varphi(d\varphi(\text{grad } \ln F)) + (m-2)(\Delta F^2)d\varphi(\text{grad } \ln F) \\
& \quad + 2(m-2)\nabla_{\text{grad } F^2}^\varphi d\varphi(\text{grad } \ln F) \\
& \quad + (m-2)\nabla_{\text{grad } \ln F}^\varphi(F^2\tau(\varphi, g) - (m-2)F^2d\varphi(\text{grad } \ln F))\} \\
& = F^4\{\tau^2(\varphi, g) + (m-2)J_g^\varphi(d\varphi(\text{grad } \ln F)) \\
& \quad + 2(\Delta \ln F - (m-4)|\text{grad } \ln F|^2)\tau(\varphi, g) - (m-6)\nabla_{\text{grad } \ln F}^\varphi\tau(\varphi, g) \\
& \quad - 2(m-2)(\Delta \ln F - (m-4)|\text{grad } \ln F|^2)d\varphi(\text{grad } \ln F) \\
& \quad + (m-2)(m-6)\nabla_{\text{grad } \ln F}^\varphi d\varphi(\text{grad } \ln F)\}.
\end{aligned}$$

This gives the second part of the theorem. \square

Corollary 1. *Let $\varphi : (M^2, g) \longrightarrow (N^n, h)$ be a map and $\bar{g} = F^{-2}g$ be a conformal change of the metric g . Let $\tau^2(\varphi, g)$ and $\tau^2(\varphi, \bar{g})$ be the bitension fields of φ with respect to the metrics g and \bar{g} respectively. Then,*

$$\begin{aligned}
(13) \quad \tau^2(\varphi, \bar{g}) &= F^4\{\tau^2(\varphi, g) + 2(\Delta \ln F + 2|\text{grad } \ln F|^2)\tau(\varphi, g) \\
&\quad + 4\nabla_{\text{grad } \ln F}^\varphi\tau(\varphi, g)\}.
\end{aligned}$$

Proof. Substituting $m = 2$ into the Equation (4) we get

$$\begin{aligned}
(14) \quad \tau^2(\varphi, \bar{g}) &= F^4\tau^2(\varphi, g) + F^2\{(\Delta F^2)\tau(\varphi, g) + 2\nabla_{\text{grad } F^2}^\varphi\tau(\varphi, g)\} \\
&= F^4\{\tau^2(\varphi, g) + 2(\Delta \ln F + 2|\text{grad } \ln F|^2)\tau(\varphi, g) + 4\nabla_{\text{grad } \ln F}^\varphi\tau(\varphi, g)\}.
\end{aligned}$$

\square

Corollary 2. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a harmonic map with $m \neq 2$, and let $\bar{g} = F^{-2}g$ be a conformal change of the metric g . Then, the map $\varphi : (M^m, \bar{g}) \longrightarrow (N^n, h)$ is a biharmonic map if and only if,*

$$\begin{aligned}
(15) \quad & J_g^\varphi(d\varphi(\text{grad } \ln F)) + (m-6)\nabla_{\text{grad } \ln F}^\varphi d\varphi(\text{grad } \ln F) \\
& - 2(\Delta \ln F - (m-4)|\text{grad } \ln F|^2)d\varphi(\text{grad } \ln F) = 0.
\end{aligned}$$

Proof. The corollary is obtained by applying Theorem 1 with $\tau(\varphi, g) = \tau^2(\varphi, g) = 0$ and $m \neq 2$. \square

Remark 1. Let $\gamma = -\ln F$, then Corollary 2 recovers Proposition 2.1 in [1] after taking into account that their convention for Laplacian on functions is $\Delta f = -\text{trace} \nabla df$ which is different from ours by a negative sign.

Example 1. The conformal immersion from Euclidean space into the hyperbolic space

$$(16) \quad \varphi : (\mathbb{R}^3 \times \mathbb{R}^+, \bar{g} = \delta_{ij}) \longrightarrow (H^5 = \mathbb{R}^4 \times \mathbb{R}^+, h = y_5^{-2} \delta_{\alpha\beta})$$

given by $\varphi(x_1, \dots, x_4) = (1, x_1, \dots, x_4)$ is a proper biharmonic map. In fact, the associated isometric immersion

$$(17) \quad \varphi : (\mathbb{R}^3 \times \mathbb{R}^+, g = x_4^{-2} \delta_{ij}) \longrightarrow (H^5 = \mathbb{R}^4 \times \mathbb{R}^+, h = y_5^{-2} \delta_{\alpha\beta})$$

is totally geodesic and hence harmonic. Here, $\bar{g} = F^{-2}g$ with $F = x_4^{-1}$. By Corollary 2, the conformal immersion (16) is biharmonic if and only if Equation (15) holds, which is equivalent to $J_{\bar{g}}^\varphi(d\varphi(\text{grad}_{\bar{g}} \ln F)) = 0$. A straightforward computation yields

$$J_{\bar{g}}^\varphi(d\varphi(\text{grad}_{\bar{g}} \ln F)) = -x_4^{-1} J_g^\varphi(d\varphi(\partial_4)) + \Delta(x_4^{-1}) d\varphi(\partial_4) + 2\nabla_{\text{grad}(x_4^{-1})}^\varphi d\varphi(\partial_4),$$

which is identically zero as one can check that

$$-x_4^{-1} J_g^\varphi(d\varphi(\partial_4)) = -4x_4^{-3} \partial y^5, \quad \Delta(x_4^{-1}) d\varphi(\partial_4) = 2x_4^{-3} \partial y^5 = 2\nabla_{\text{grad}(x_4^{-1})}^\varphi d\varphi(\partial_4).$$

Example 2. The conformal immersion from Euclidean space into the sphere

$$\varphi : (\mathbb{R}^4, \bar{g} = \delta_{ij}) \longrightarrow (S^5 \setminus \{N\} \equiv \mathbb{R}^5, h = \frac{4\delta_{\alpha\beta}}{(1 + |y|^2)^2})$$

given by $\varphi(u_1, \dots, u_4) = (u_1, \dots, u_4, 0)$, where (u_1, \dots, u_5) are conformal coordinates on $S^5 \setminus \{N\} \equiv \mathbb{R}^5$, is a proper biharmonic map. In fact, the map is the inverse stereographic projection that maps \mathbb{R}^4 into a great hypersphere in S^5 . The associated isometric immersion

$$\varphi : (S^4 \setminus \{P\} \equiv \mathbb{R}^4, \bar{g} = \frac{4\delta_{ij}}{(1 + |u|^2)^2}) \longrightarrow (S^5 \setminus \{N\} \equiv \mathbb{R}^5, h = \frac{4\delta_{\alpha\beta}}{(1 + |y|^2)^2})$$

is totally geodesic and hence harmonic. A computation similar to those in Example 1 shows that the conformal immersion is indeed a proper biharmonic map.

3. CONFORMAL BIHARMONIC IMMERSIONS

Proposition 1. Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a conformal immersion with $\varphi^*h = \lambda^2 g$. Let $\varphi : (M^m, \bar{g}) \longrightarrow (N^n, h)$ be the associated isometric immersion

with mean curvature vector η , where $\bar{g} = \varphi^*h = \lambda^2g$. Then, the conformal immersion $\varphi : (M^m, g) \longrightarrow (N^n, h)$ is biharmonic if and only if

$$(18) \quad \begin{aligned} \lambda^4 \tau^2(\varphi, \bar{g}) &= -(m-2)J_g^\varphi(d\varphi(\text{grad } \ln \lambda)) + 2m\lambda^2(-\Delta \ln \lambda - 2|\text{grad } \ln \lambda|^2)\eta \\ &+ m(m-6)\lambda^2 \nabla_{\text{grad } \ln \lambda}^\varphi \eta. \end{aligned}$$

Proof. Substituting $F = \lambda^{-1}$ and $\ln F = -\ln \lambda$ into the Equation (4) we have

$$(19) \quad \begin{aligned} \tau^2(\varphi, \bar{g}) &= \lambda^{-4} \{ \tau^2(\varphi, g) - (m-2)J_g^\varphi(d\varphi(\text{grad } \ln \lambda)) \\ &+ 2(-\Delta \ln \lambda - (m-4)|\text{grad } \ln \lambda|^2)\tau(\varphi, g) + (m-6)\nabla_{\text{grad } \ln \lambda}^\varphi \tau(\varphi, g) \\ &+ 2(m-2)(-\Delta \ln \lambda - (m-4)|\text{grad } \ln \lambda|^2)d\varphi(\text{grad } \ln \lambda) \\ &+ (m-2)(m-6)\nabla_{\text{grad } \ln \lambda}^\varphi d\varphi(\text{grad } \ln \lambda) \}. \end{aligned}$$

Note that the tension field of the conformal immersion φ is given by

$$(20) \quad \tau(\varphi) = m\lambda^2\eta + (2-m)d\varphi(\text{grad } \ln \lambda).$$

Substituting (20) into (19) we have

$$(21) \quad \begin{aligned} \tau^2(\varphi, \bar{g}) &= \lambda^{-4} \{ \tau^2(\varphi, g) - (m-2)J_g^\varphi(d\varphi(\text{grad } \ln \lambda)) \\ &+ 2m\lambda^2(-\Delta \ln \lambda - 2|\text{grad } \ln \lambda|^2)\eta \\ &+ m(m-6)\lambda^2 \nabla_{\text{grad } \ln \lambda}^\varphi \eta \}. \end{aligned}$$

From this we obtain the proposition. \square

Theorem 2. *The conformal immersion $\varphi : (M^2, g) \longrightarrow (\mathbb{R}^3, \langle \cdot, \cdot \rangle_0)$ into Euclidean space with $\varphi^*\langle \cdot, \cdot \rangle_0 = \lambda^2g$ is biharmonic if and only if*

$$(22) \quad \begin{cases} A_\xi(\text{grad } H) + \frac{1}{2}\text{grad}(H^2) + 2H A_\xi(\text{grad } \ln \lambda) = 0 \\ \Delta H - H|B|^2 + 2H(\Delta \ln \lambda + 2|\text{grad } \ln \lambda|^2) + 4g(\text{grad } \ln \lambda, \text{grad } H) = 0, \end{cases}$$

where ξ is the unit normal vector field of the surface $\varphi(M) \subset \mathbb{R}^3$ and A_ξ and H are the shape operator and the mean curvature function of the surface respectively.

Proof. It follows from (18) with $m = 2$ that the conformal immersion φ is biharmonic if and only if

$$(23) \quad \lambda^2 \tau^2(\varphi, \bar{g}) = -4(\Delta \ln \lambda + 2|\text{grad } \ln \lambda|^2)\eta - 8\nabla_{\text{grad } \ln \lambda}^\varphi \eta,$$

where $\tau^2(\varphi, \bar{g})$ denotes the bitension field of the associated isometric immersion $\varphi : (M^2, \bar{g} = \lambda^2g) \longrightarrow \mathbb{R}^3$ with mean curvature vector $\eta = H\xi$, where ξ and H are the unit normal vector field and the mean curvature function of the surface $\varphi(M)$ respectively. Then, we have (see, e.g., [11], [6] and [5])

$$\tau^2(\varphi, \bar{g}) = 2(\Delta_{\bar{g}}H - H|B|_{\bar{g}}^2)\xi - 2[2A_\xi(\text{grad}_{\bar{g}}H) + \text{grad}_{\bar{g}}(H^2)],$$

substitute this into (23) we have

$$(24) \quad \begin{aligned} & \lambda^2(\Delta_{\bar{g}}H - H|B|^2)\xi - \lambda^2[2A_\xi(\text{grad}_{\bar{g}}H) + \text{grad}_{\bar{g}}(H^2)] \\ &= -2(\Delta \ln \lambda + 2|\text{grad} \ln \lambda|^2)H\xi - 4\nabla_{\text{grad} \ln \lambda}^\varphi H\xi. \end{aligned}$$

Notice that the transformations of Laplacian and the gradient operators under a conformal change of metrics $\bar{g} = \lambda^2 g$ in two dimensional manifold are given by

$$(25) \quad \Delta_{\bar{g}}u = \lambda^{-2}\Delta u, \quad \text{grad}_{\bar{g}}u = \lambda^{-2}\text{grad}u.$$

On the other hand, we have

$$(26) \quad -4\nabla_{\text{grad} \ln \lambda}^\varphi H\xi = -4g(\text{grad} \ln \lambda, \text{grad} H)\xi + 4H A_\xi(\text{grad} \ln \lambda).$$

Using (25) and substituting (26) into (24) and comparing the tangential and the normal components we obtain equation (22) which completes the proof of the theorem. \square

Proposition 2. *For $\lambda^2 = (C_2 e^{\pm z/R} - C_1 C_2^{-1} R^2 e^{\mp z/R})/2$ with constants C_1, C_2 , the maps $\phi : (D, g = \lambda^{-2}(R^2 d\theta^2 + dz^2)) \longrightarrow (\mathbb{R}^3, d\sigma^2 = d\rho^2 + \rho^2 d\theta^2 + dz^2)$ with $\phi(\theta, z) = (R, \theta, z)$ is a family of proper biharmonic conformal immersions of a cylinder of radius R into Euclidean space \mathbb{R}^3 .*

Proof. Let $\phi : \mathbb{R}^2 \supseteq D \longrightarrow \mathbb{R}^3$, $\phi(\theta, z) = (R \cos \theta, R \sin \theta, z)$ be the isometric immersion with the image $\phi(D)$ being a cylinder of radius R in 3-space. Using cylindrical coordinates (ρ, θ, z) on \mathbb{R}^3 we can represent the isometric immersion of the cylinder as $\phi : \mathbb{R}^2 \supseteq D \longrightarrow \mathbb{R}^3$ with $\phi(\theta, z) = (R, \theta, z)$. It is easy to check that $E_1 = \frac{\partial}{\partial \rho}$, $E_2 = \frac{1}{\rho} \frac{\partial}{\partial \theta}$, $E_3 = \frac{\partial}{\partial z}$ constitute a local orthonormal frame of \mathbb{R}^3 and that $e_1 = E_2$, $e_2 = E_3$, $\xi = E_1$ is an adapted orthonormal frame along the cylinder with ξ being unit normal vector field. We can check that the induced metric on the cylinder is $\bar{g} = R^2 d\theta^2 + dz^2$. Let $g = \lambda^{-2}(R^2 d\theta^2 + dz^2)$ be a conformal change of the metric on the cylinder. Then, we have a conformal immersion $\phi : (D, g) \longrightarrow (\mathbb{R}^3, d\sigma^2 = d\rho^2 + \rho^2 d\theta^2 + dz^2)$ with $\phi^* d\sigma^2 = \lambda^2 g = \bar{g}$. A straightforward computation gives

$$(27) \quad \begin{cases} A_\xi e_1 = -\frac{1}{R}e_1, & A_\xi e_2 = 0, \\ H = \frac{1}{2}(\langle A_\xi e_1, e_1 \rangle + \langle A_\xi e_2, e_2 \rangle) = -\frac{1}{2R} \neq 0 \\ |B|^2 = \lambda^2 |B|_{\bar{g}}^2 = \lambda^2 \sum_{i=1}^2 |A(e_i)|^2 = \lambda^2 \frac{1}{R^2}, \\ \text{grad} H = 0, \\ \Delta H = 0. \end{cases}$$

Substituting (27) into (22) we conclude that conformal immersion ϕ is biharmonic if and only if

$$\begin{cases} A_\xi(\text{grad } \ln \lambda) = 0 \\ \lambda^2 - 2R^2(\Delta \ln \lambda + 2|\text{grad } \ln \lambda|^2) = 0. \end{cases}$$

It is not difficult to check that this system is equivalent to

$$\begin{cases} \lambda(\theta, z) = \lambda(z) \\ 1 - 2R^2[(\ln \lambda)'' + 2(\ln \lambda)'^2] = 0 \end{cases}$$

or,

$$(\lambda^2)'' = \frac{1}{R^2} \lambda^2.$$

It follows that λ^2 is a solution of the ordinary differential equation

$$y'' = \frac{1}{R^2} y,$$

which has (see e.g., [8]) the first integral

$$(28) \quad y'^2 = y^2/R^2 + C_1.$$

Solving Equation (28) we have

$$(29) \quad y = (C_2 e^{\pm z/R} - C_1 C_2^{-1} R^2 e^{\mp z/R})/2.$$

Notice that the conformal immersion has nonzero constant mean curvature H so it is not harmonic. Therefore, we complete the proof of the proposition. \square

Remark 1. It follows from Proposition 2 that the biharmonic conformal immersions of the cylinder in \mathbb{R}^3 are not minimal, thus the well-known fact that a conformal harmonic immersion of a surface must be a minimal surface fails to generalize to conformal biharmonic immersion of a surface. Our proposition also shows that if B. Y. Chen's conjecture [6] about biharmonic isometric immersions into Euclidean space is generalized to biharmonic conformal immersions, then the answer is negative.

4. BIHARMONICITY AND HOLOMORPHICITY OF CONFORMAL IMMERSIONS

Let $\varphi : (M^2, g) \longrightarrow \mathbb{R}^n$ be a conformal immersion of a Riemann surface. Let (u, v) be the local coordinates on M and we write $z = u + iv$ in the local complex parameter. We also use the usual notations

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right), \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right).$$

Then, the well-known Weierstrass representation theorem for conformal harmonic immersions can be stated as: Let $\varphi : (M^2, g) \longrightarrow (\mathbb{R}^n, \langle \cdot, \cdot \rangle_0)$ be a harmonic

conformal immersion. Then, the section $\phi = \frac{\partial \varphi}{\partial z} = \frac{1}{2}(\varphi_u - i\varphi_v) = \phi^\alpha(z) \frac{\partial}{\partial y^\alpha}$ is holomorphic and satisfies

$$(30) \quad \sum_{\alpha=1}^n (\phi^\alpha)^2 = 0,$$

$$(31) \quad \sum_{\alpha=1}^n |\phi^\alpha|^2 \neq 0.$$

Conversely, given any holomorphic section $\phi = \phi^\alpha \frac{\partial}{\partial y^\alpha} : M \longrightarrow \mathbb{E}$ satisfying (30) and (31) and the periodic condition:

$$\Re \int_{\gamma} (\phi^1, \dots, \phi^n) dz = 0,$$

for any closed path in M . Then, the map

$$\varphi(z) = 2 \Re \int_{z_0}^z (\phi^1, \dots, \phi^n) dz$$

defines a harmonic conformal immersion of a Riemann surface into the Euclidean space.

For conformal biharmonic immersions of surfaces into Euclidean space, we have

Theorem 3. $\varphi : (M^2, g) \longrightarrow (\mathbb{R}^n, \langle \cdot, \cdot \rangle_0)$ is a conformal biharmonic immersion with $\varphi^* \langle \cdot, \cdot \rangle = \lambda^2 g$ if and only if the section $\phi = \frac{\partial \varphi}{\partial z} = \frac{1}{2}(\varphi_u - i\varphi_v) = \phi^\alpha(z) \frac{\partial}{\partial y^\alpha}$ satisfies Equations (30), (31) and

$$(32) \quad \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z} \left(\lambda^{-2} \frac{\partial \phi^\sigma}{\partial \bar{z}} \right) = 0$$

Proof. Using the local conformal parameter $z = u + iv$ and the characteristic property that $g = \varphi^* \langle \cdot, \cdot \rangle_0$ of the conformal immersion $\varphi : (M^2, g) \longrightarrow (\mathbb{R}^n, \langle \cdot, \cdot \rangle_0)$ we can write the metric g as $g = \lambda^2(du^2 + dv^2) = \lambda^2|dz|^2$, where $\lambda^2 = \langle \varphi_u, \varphi_u \rangle_0 = \langle \varphi_v, \varphi_v \rangle_0$. The Laplacian operator on (M, g) can be written as

$$\Delta = \lambda^{-2} \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) = 4\lambda^{-2} \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z}.$$

Let $\{y^\alpha, \frac{\partial}{\partial y^\alpha}\}$ be local coordinates in a neighborhood U of \mathbb{R}^n such that $U \cap \varphi(M)$ is nonempty. We can write the local expression $\varphi(z) = (\varphi^1(z), \dots, \varphi^n(z))$ as $\varphi(z) = \varphi^\alpha(z) \frac{\partial}{\partial y^\alpha}$. If we define the section $\phi = \frac{\partial \varphi}{\partial z} = \frac{1}{2}(\varphi_u - i\varphi_v)$. Then, it is well-known (see, e.g., [9]) the tension field of φ can be written as

$$\tau(\varphi) = (\Delta \varphi^1, \dots, \Delta \varphi^n) = 4\lambda^{-2} \frac{\partial^2 \varphi^\sigma}{\partial \bar{z} \partial z} \frac{\partial}{\partial y^\sigma} = 4\lambda^{-2} \frac{\partial \phi^\sigma}{\partial \bar{z}} \frac{\partial}{\partial y^\sigma},$$

and hence the bitension field is given by

$$(33) \quad \tau^2(\varphi) = (\Delta^2 \varphi^1, \dots, \Delta^2 \varphi^n) = 4\lambda^{-2} \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z} \left(4\lambda^{-2} \frac{\partial \phi^\sigma}{\partial \bar{z}} \right) \frac{\partial}{\partial y^\sigma}.$$

It is easy to see that Equations (30) and (31) is equivalent to φ being a conformal immersion whilst Equation (32) is equivalent to the biharmonicity of φ by (33). \square

Example 3. We can use Weierstrass representation to prove that the map $\varphi : (\mathbb{R}^2, g = e^{y/R}(dx^2 + dy^2)) \longrightarrow \mathbb{R}^3$, $\varphi(x, y) = (R \cos \frac{x}{R}, R \sin \frac{x}{R}, y)$ is a proper biharmonic conformal immersion of \mathbb{R}^2 into Euclidean space \mathbb{R}^3 .

Indeed, in this case, $\varphi_x = (-\sin \frac{x}{R}, \cos \frac{x}{R}, 0)$, $\varphi_y = (0, 0, 1)$ and φ is a conformal immersion with $\varphi^* \langle \cdot, \cdot \rangle_0 = \bar{g} = dx^2 + dy^2 = \lambda^2 g$ for $\lambda^2 = e^{-y/R}$. The section $\phi = \frac{\partial \varphi}{\partial z} = \frac{1}{2}(\varphi_x - i\varphi_y) = \frac{1}{2}(-\sin \frac{x}{R}, \cos \frac{x}{R}, -i)$ with components

$$(34) \quad \phi^1 = -\frac{1}{2} \sin \frac{z + \bar{z}}{2R}, \quad \phi^2 = \frac{1}{2} \cos \frac{x}{R} = \frac{1}{2} \cos \frac{z + \bar{z}}{2R}, \quad \phi^3 = -i/2.$$

A straightforward computation yields

$$(35) \quad \begin{cases} \lambda^{-2} \frac{\partial \phi^1}{\partial \bar{z}} = -\frac{1}{4R} e^{-(z-\bar{z})i/(2R)} \cos \frac{z+\bar{z}}{2R} = -\frac{i}{8R} (e^{\bar{z}i/R} + e^{-zi/R}), \\ \lambda^{-2} \frac{\partial \phi^2}{\partial \bar{z}} = -\frac{1}{4R} e^{-(z-\bar{z})i/(2R)} \sin \frac{z+\bar{z}}{2R} = -\frac{i}{8R} (e^{e^{-zi/R} - \bar{z}i/R}), \\ \lambda^{-2} \frac{\partial \phi^3}{\partial \bar{z}} = 0. \end{cases}$$

Clearly, we have $\frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z} (\lambda^{-2} \frac{\partial \phi^\sigma}{\partial \bar{z}}) = 0$ for $\sigma = 1, 2, 3$ and hence, by Theorem 3, φ is a biharmonic conformal immersion which is not harmonic as the section ϕ is not holomorphic.

Example 4. We can also easily check that the map $\varphi : (\mathbb{R}^2, g = e^{y/R}(dx^2 + dy^2)) \longrightarrow \mathbb{R}^6$, $\varphi(x, y) = (R \cos \frac{x}{R}, R \sin \frac{x}{R}, y, R \cos \frac{x}{R}, R \sin \frac{x}{R}, y)$ is a proper biharmonic conformal immersion of \mathbb{R}^2 into Euclidean space \mathbb{R}^6 .

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